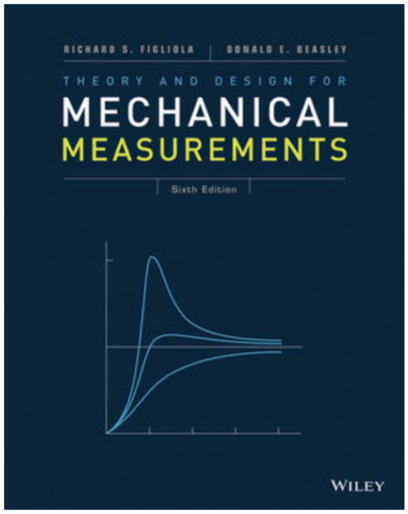


Fourier Series and Fourier Transforms

Engr325

Instrumentation

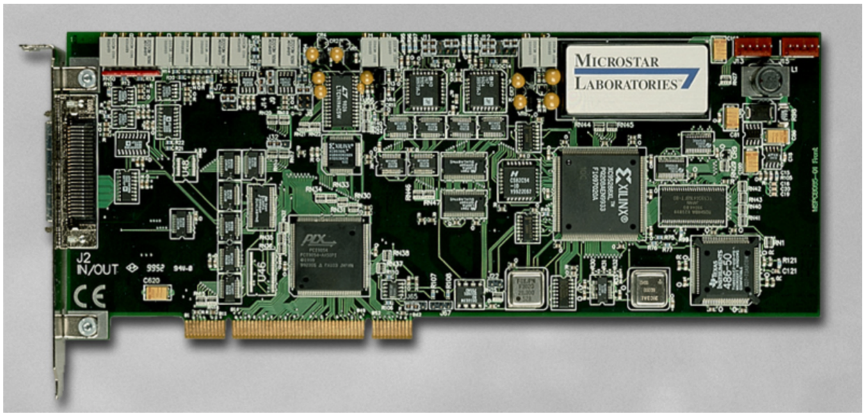
Dr Curtis Nelson

This Lecture

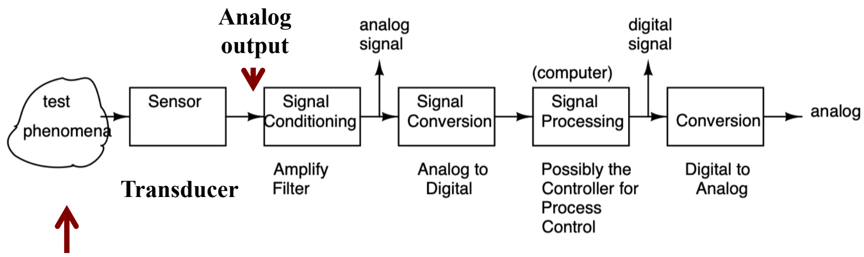
- Fourier series.
- Fourier transforms.

A/D Converter Board

http://www.mstarlabs.com/dataacquisition/840/840spec.html



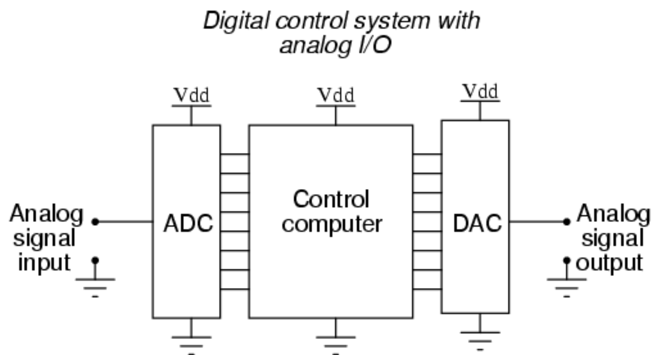
Instrumentation System



Temperature, Light, Displacement,
Position, Vibration, Strain, Angle,
Velocity, Acceleration, Flow rate,
Pressure, Viscosity, Heart Rate, Etc.

Digital Acquisition System

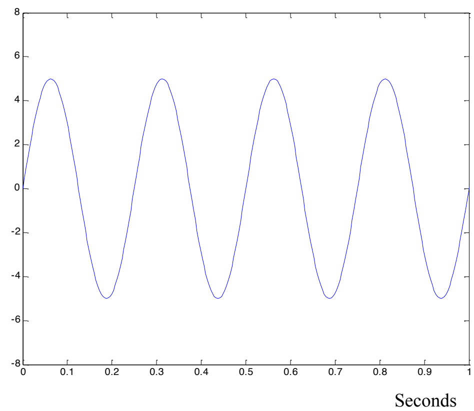
- Computers are nearly always in the middle of any instrumentation system to provide a complete interface with analog sensors and output devices.



Signal Processing

- The big question now is: “What information do we want to extract from the signal that we have acquired?”
  - Amplitude.
  - Frequency.
  - Timing.
  - Phase.
  - Etc.
- Let’s focus on **Frequency** today.

A Sine Wave

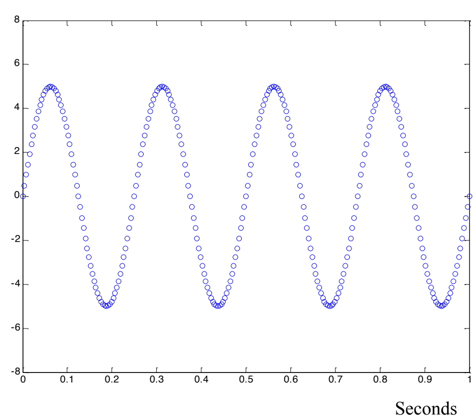


$5*\sin(2\pi 4t)$

Amplitude = 5

Frequency = 4 Hz

A Sampled Sine Wave Signal



$5*\sin(2\pi 4t)$

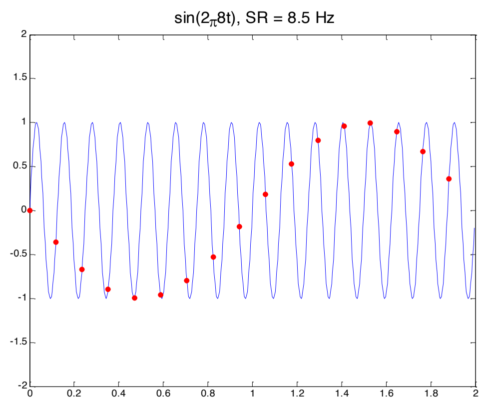
Amplitude = 5

Frequency = 4 Hz

Sampling rate = 256 samples/second

Sampling duration = 1 second

An Undersampled Signal



The Nyquist Frequency

- The Nyquist frequency is equal to one-half of the sampling frequency.
- The Nyquist frequency is the highest frequency that can be measured in a signal.
- In other words – you **MUST** sample at least twice as fast as the fastest frequency in the signal you wish to capture.
- And, how do you determine what that maximum frequency is? Or more generally, how do you determine all of the frequencies comprising the signal you are capturing?
- We will answer the last question by going in reverse, i.e. we will learn how to create a signal composed of multiple frequencies.

Fourier Series

- **Fourier Series** (English pronunciation: /ˈfɔəriɪ/) is a way to represent a wave-like function as the sum of simple sine waves. More formally, it decomposes any *periodic function* or periodic signal into the sum of a (possibly infinite) set of sine and cosine functions.

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} y(t)dt$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos n\omega t dt \qquad (2.13)$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin n\omega t dt$$

where $n = 1, 2, 3, \ldots$, and $T = 2\pi/\omega$ is the period of $y(t)$. The trigonometric series that results from these coefficients is a Fourier series and may be written as

$$y(t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\omega t + B_n \sin n\omega t) \qquad (2.14)$$

Alternative Mathematical Representations

$$y(t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\omega t \; + \; B_n \sin n\omega t)$$

may be written as

$$y(t) = A_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega t - \phi_n\right)$$

or

$$y(t) = A_0 + \sum_{n=1}^{\infty} C_n \sin\left(n\omega t + \phi_n^*\right)$$

where

$$C_n = \sqrt{A_n^2 + B_n^2}$$

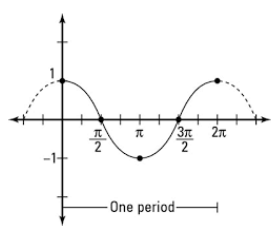
$$\tan \phi_n = \frac{B_n}{A_n} \quad \text{and} \quad \tan \phi_n^* = \frac{A_n}{B_n}$$

Fourier Series

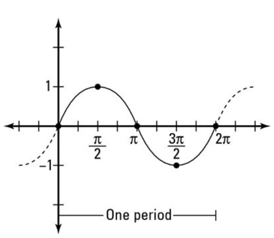
- Introduction
  - https://www.youtube.com/watch?v=UKHBWzoOKsY
- Frequency filtering
  - https://www.youtube.com/watch?v=JndvN1ngSi4
- Matlab demo

Fourier Series Symmetry

- If *f(t)* = *f(-t)*, then *f(t)* is said to be an **even** function and all the b_n terms = 0. In other words, *f(t)* is mirrored about the y-axis, like the cosine function.

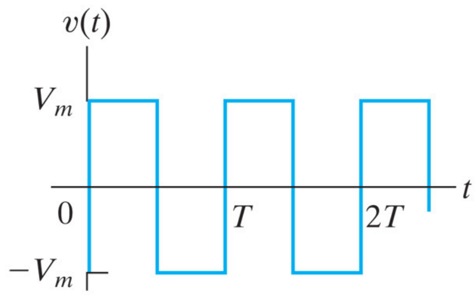


- If *f(t)* = -*f(-t)*, then *f(t)* is said to be an **odd** function and all a_n terms (including the dc value) = 0. The sine function is an example.



Fourier Series Example

- Find the Fourier Series for the following waveform. Set $V_m = 1$ and $T = 2\pi$. See the Matlab file *fourier_series_square.m* on the course web page for implementation.



$$v(t) = \sum_{n=1}^{N} \frac{4}{n\pi} \sin(nt)$$

The Fourier Transform

- A transform takes one function (or signal) and turns it into another function (or signal).
- Continuous Fourier and Inverse Fourier Transforms:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t} df$$

- Note that the transforms contain complex numbers.

The Discrete Fourier Transform

- Because we have computers in the mix and deal with discrete samples of information, mathematicians developed the **Discrete Fourier Transform** (*DFT*):

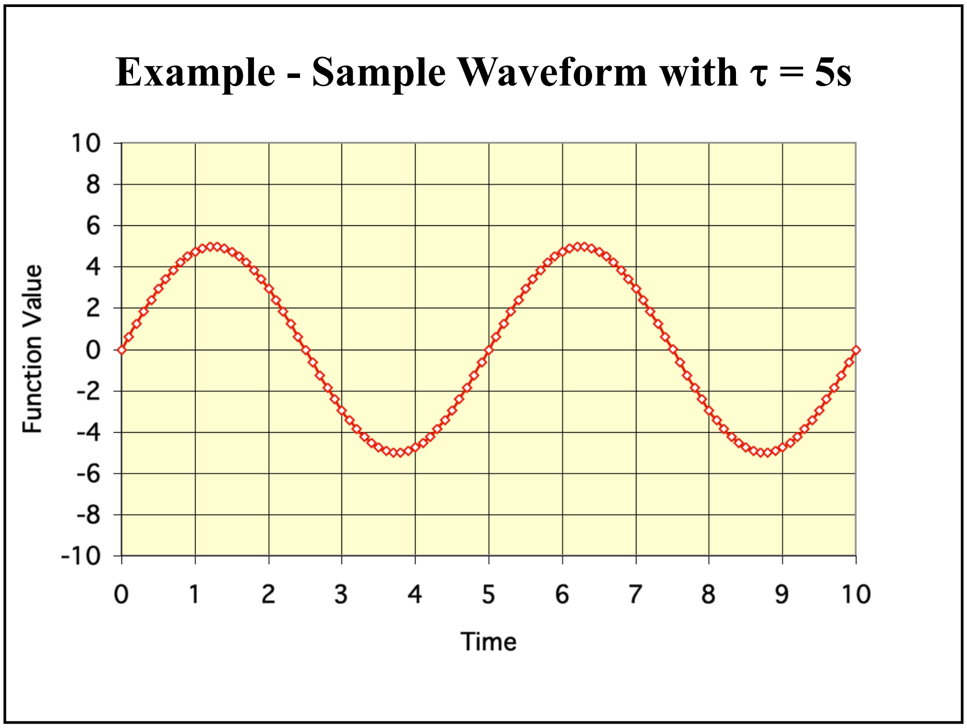
$$H_n = \sum_{k=0}^{N-1} h_k e^{2\pi ikn/N}$$

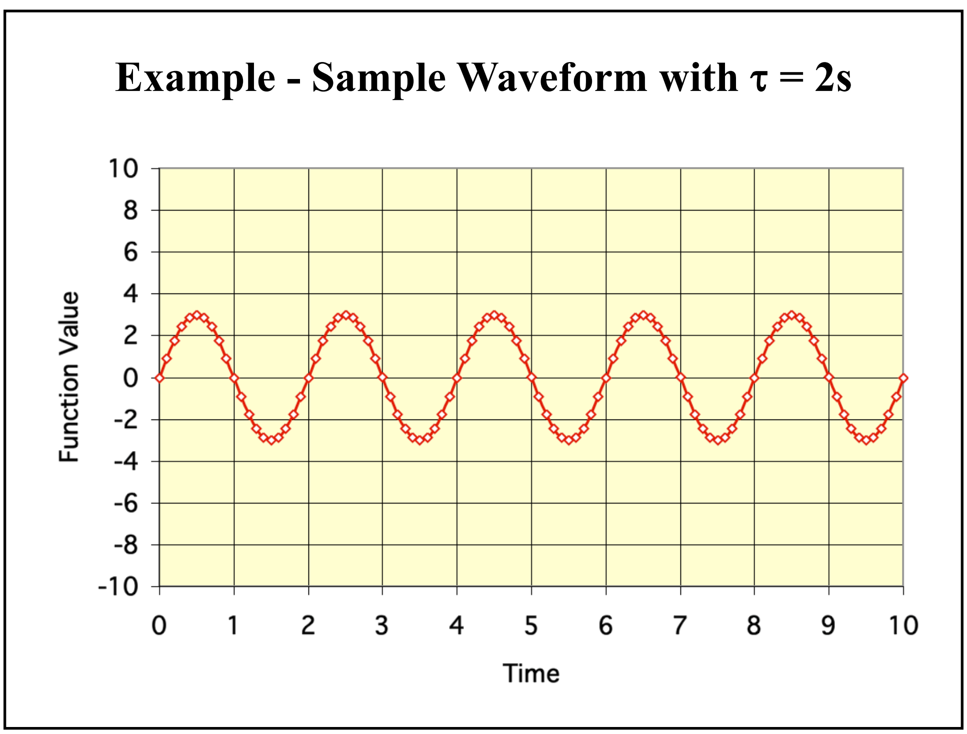
$$h_k = \frac{1}{N}\sum_{n=0}^{N-1} H_n e^{-2\pi ikn/N}$$

- The *DFT* is also complex in nature.

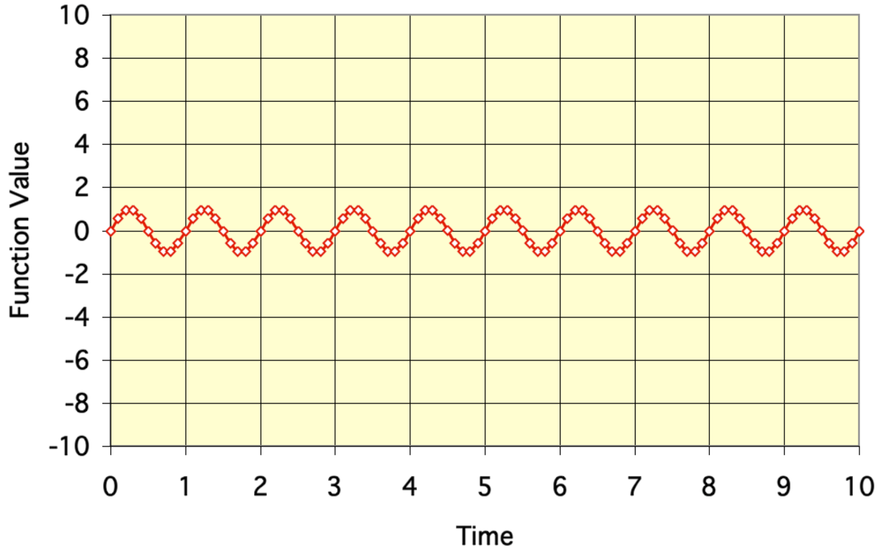
Fast Fourier Transform

- You have probably heard about the The **Fast Fourier Transform** (*FFT*). The *FFT* is an efficient algorithm for performing a Discrete Fourier Transform.
- The FFT algorithm was first published by Cooley & Tukey in 1965.
- In 1969, the 2048 point analysis of a seismic data trace took over 13 hours. Using the FFT, the same task on the same machine took 2.4 seconds.
- The FFT is, by far, the most common frequency analysis algorithm used in programs such as Matlab.

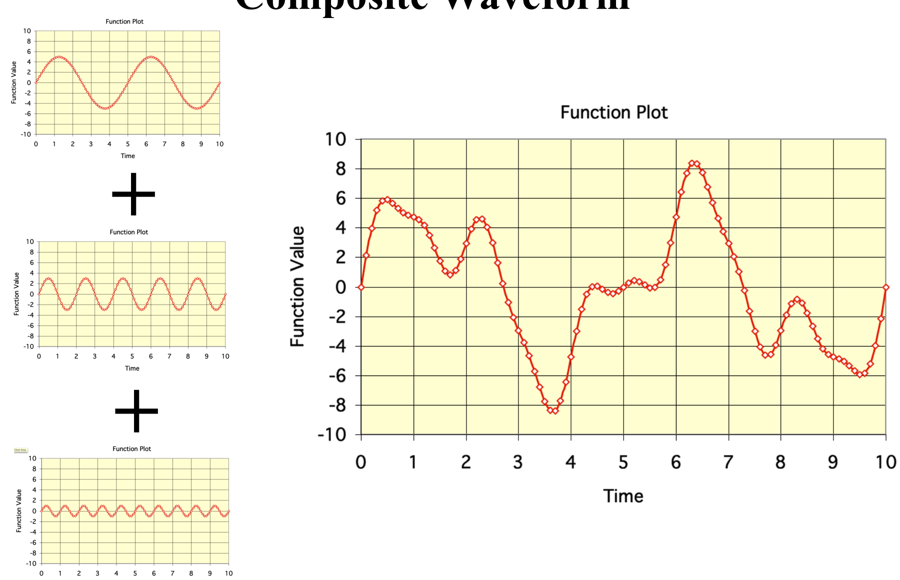

Example - Sample Waveform with τ = 5s
Function Value
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
6
7
8
9
10
Time


Example - Sample Waveform with τ = 2s
Function Value
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
6
7
8
9
10
Time

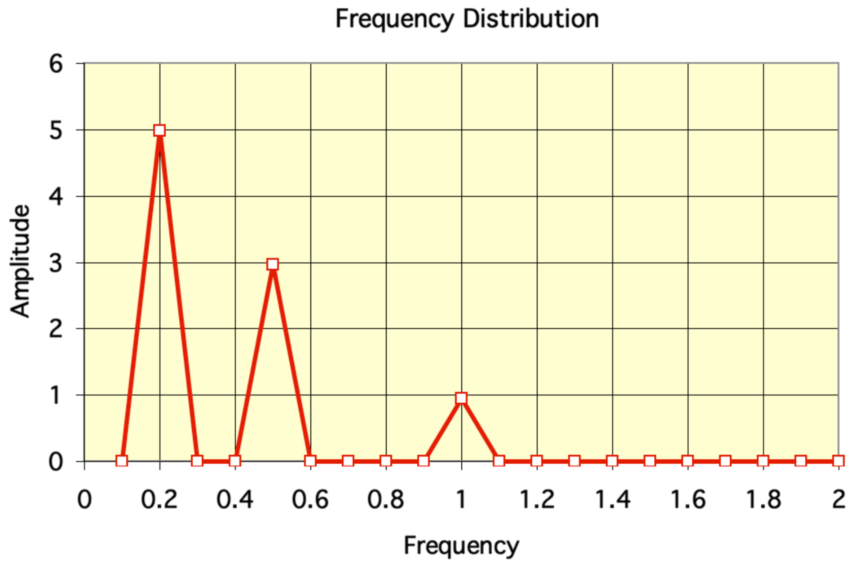
Example - Sample Waveform with τ = 1s



Composite Waveform



Fourier Transform Information



A Little Video Help

- [Graphical Depiction of Fourier Series (watch first 10 minutes)]
- [Introduction to Fourier Transform]

Notes on the Fourier Transform

- The FFT takes, as inputs, a vector of discrete points representing magnitudes in the time domain.
- The FFT returns a set of complex numbers containing magnitude and phase information in the frequency domain.
  - Note that the data is duplicated from the midpoint on, actually mirrored about the x-axis since it returns values for frequencies between –infinity and +infinity.
- Sampling rate issues
  - Sampling too slow.
  - Sampling too fast
    - Possibility of not acquiring an entire period of the lowest frequencies.
- FFT illustration via Matlab.

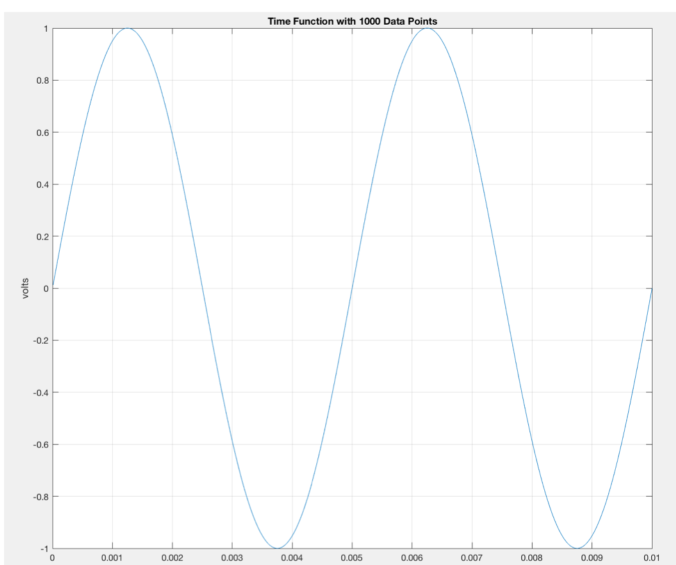
Matlab Code

```
% FFT Example
% Revision 3.0 Curt Nelson  1/28/2020
% Creates functions of time and explores various fft implementations
clear all;
clear plot;

% Calculate the number of time samples (1000 in this case).
start_time = 0;
end_time = .01;                % 10 milli-seconds
delta_time = 1e-5;             % 10 micro-seconds
number_time_samples = (end_time - start_time)/delta_time;  % 1000 points

% Create the time vector for x-axis (1000 data points)
time_vec = start_time:delta_time:end_time;
```

Matlab Code

```
% The sampling rate is 1/delta_time or 100,000 samples/second
sampling_rate = 1/delta_time;

% Next create the time domain function with a frequency of 200Hz, resulting
% in a period of 1/frequency or 5 milli-seconds.
freq = 200;
period = 1/freq;
time_function = sin(2*pi*freq*time_vec);

% Since we are sampling from 0 to 10ms, we should see 2 cycles
plot(time_vec,time_function);
title('Time Function with 1000 Data Points');
ylabel('volts');
xlabel('time - seconds');
grid on;
pause;
```

time_function = sin(2*pi*200*x1t)



Matlab Code

```
% Now do an FFT on this time domain function
% fft_results contain a complex number pair for each sample
fft_results = fft(time_function);

% Create the x axis for frequencies starting at the DC value (0 Hz)
dc_value = 0;

% We only need to plot the first half of the frequencies because the fft returns
% the same data folded over on itself at maxfreq/2
% Frequency spacing is the sampling rate / by the number of samples and is
% the frequency resolution on the x axis.
freq_spacing =  sampling_rate/number_time_samples;

% Maximum frequency for the fft is (sampling rate/2) – freq_spacing
freq_max = (sampling_rate/2) - freq_spacing;
```

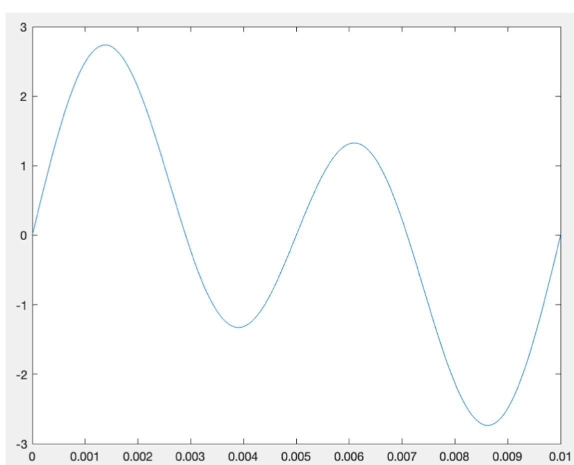
FFT Matlab Code

```
% Next, create the x-axis points (0 - 49,900 in increments of 100Hz)
freq_plot_xaxis = dc_value:freq_spacing:freq_max;

% This results in (number of time samples/2) or 500 frequencies
number_freq_samples = number_time_samples/2;

% The magnitude of the fft must be computed from the complex fft_results
magnitude = abs(fft_results);

% Normalize magnitude by dividing by the number of frequency samples
nor_magnitude = magnitude/number_freq_samples;

% Plot the first 30 frequencies using red circles
plot(freq_plot_xaxis(1:30),nor_magnitude(1:30),'ro');
pause;
```

FFT of sin(2*pi*200*t)

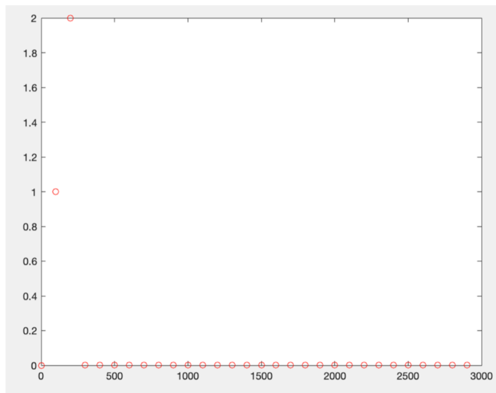
FFT Matlab Code

```
% Next, create a signal of two frequencies with different magnitudes
new_sig = sin(2*pi*100*time_vec) + 2*sin(2*pi*200*time_vec);
plot(time_vec,new_sig);
pause;
```



FFT of sin(2*pi*100*time_vec) + 2*sin(2*pi*200*time_vec)

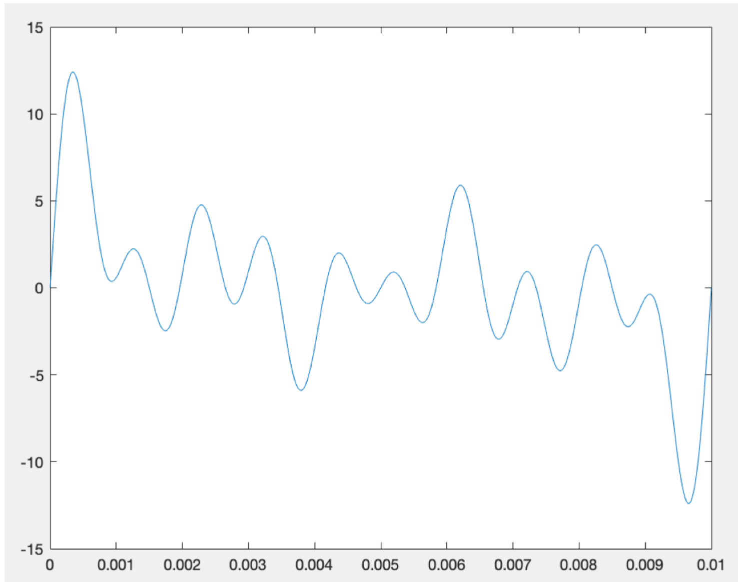
```
% Perform the fft and plot
fft_results = fft(new_sig);
magnitude = abs(fft_results);
nor_magnitude = magnitude/number_freq_samples;
plot(freq_plot_xaxis(1:30),nor_magnitude(1:30),'ro');
pause;
```



FFT Matlab Code

```
% Finally, lets add eight more frequencies with various magnitudes and plot
new_sig = new_sig + .5*sin(2*pi*300*time_vec);
new_sig = new_sig + 2*sin(2*pi*400*time_vec);
new_sig = new_sig + 3*sin(2*pi*500*time_vec);
new_sig = new_sig + sin(2*pi*600*time_vec);
new_sig = new_sig + 2*sin(2*pi*700*time_vec);
new_sig = new_sig + .5*sin(2*pi*800*time_vec);
new_sig = new_sig + sin(2*pi*900*time_vec);
new_sig = new_sig + 3*sin(2*pi*1000*time_vec);

plot(time_vec,new_sig);                % Plot the function
pause;
```

f(t) Composed of 10 Frequencies



Matlab Code

```
% Perform the fft and plot
fft_results = fft(new_sig);
magnitude = abs(fft_results);
nor_magnitude = magnitude/number_freq_samples;
plot(freq_plot_xaxis(1:30),nor_magnitude(1:30),'ro');
pause;
```